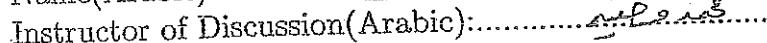
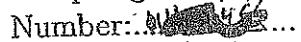


SecondBirzeit University- Mathematics Department
Calculus II-Math 132

Second Exam

Name(Arabic): Instructor of Discussion(Arabic): 

Spring 2011/2012

Number: 

Section: 14

Time: 90 Min. Calculators are not allowed. There are 4 questions in 7 pages.

Question 1.(51%) Circle the correct answer:

1. The sequence
- $a_n = (1 + \frac{1}{n})^{-n}$
- ,
- $n = 1, 2, 3, \dots$

$$\frac{1}{(1 + \frac{1}{n})^n} = \frac{1}{e^1} = e^{-1}$$

33
08
02
09
52

- (a) Converges to 1.
- (b) Converges to e.
- (c) Converges to e^{-1} .
- (d) Diverges.

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

$$\left| \frac{(-1)^{n+1}}{\sqrt{n}} \right| = \frac{1}{\sqrt{n}} \text{ div by P-series}$$

$\frac{(-1)^{n+1}}{\sqrt{n}} \rightarrow u_n \text{ decreases positive}$

$$\lim \frac{1}{\sqrt{n}} = 0$$

- (a) Converges conditionally.
- (b) Converges absolutely.
- (c) Converges by nth term test.
- (d) Diverges.

3. $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

$$\begin{aligned} &> \sqrt{n} \\ n + \sqrt{n} &> n \end{aligned}$$

- (a) Diverges by nth term test.

- (b) Converges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{2n}$.

$$\frac{1}{n+\sqrt{n}} = \frac{1}{\infty+\infty} = \frac{1}{\infty} = 0$$

- (c) Diverges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$.

- (d) Diverges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{2n}$.

4. $\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+1}\right)^n$

- (a) Converges by nth root test.

- (b) Diverges by nth root test.

- (c) Converges by alternating series test.

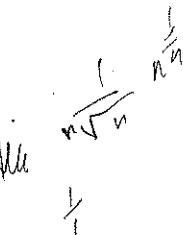
- (d) None of the above.

5. One of the following statements is always true

- (a) A bounded sequence always converges. $\times \rightarrow (-1, 1) \rightarrow (1, 1)$
- (b) A monotonic sequence converges. \times monotonic + bounded $a_n = n$
- (c) A convergent sequence is monotonic. $\times (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$
- (d) A convergent sequence is bounded.

6. The series $\sum_{n=1}^{\infty} \frac{1}{n^{1/n}}$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n}} = \frac{1}{\sqrt[1]{1}} = 1$$



- (a) Converges by n th term test.
- (b) Converges by n th root test.
- (c) Diverges by n th term test.
- (d) Diverges by n th root test.

7. The series $\sum_{n=1}^{\infty} (\log_2 x)^n$ converges if

$$\left(\frac{\ln x}{\ln 2}\right)^n = \left|\frac{\ln x}{\ln 2}\right| =$$

$$-1 < \frac{\ln x}{\ln 2} < 1$$

$$-\ln 2 < \ln x < \ln 2$$

$$e^{-\ln 2} < x < e^{\ln 2}$$

8. One of the following statements is true

- (a) If $\lim_{n \rightarrow \infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges. \times
- (b) If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 0$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge. \times
- (c) The alternating harmonic series diverges. \times
- (d) If $\lim_{n \rightarrow \infty} a_n = 1$ then $\sum_{n=1}^{\infty} a_n$ diverges. \checkmark nth term test

9. The radius of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ is

- (a) $R = 1$.
- (b) $R = 2$.
- (c) $R = 0$.
- (d) $R = \infty$.

$$\sqrt[n]{\left(\frac{x}{2}\right)^n} \rightarrow \left(\frac{x}{2}\right)^n \rightarrow \left(\frac{x}{2}\right) < 1$$

$$-1 < \frac{x}{2} < 1 \rightarrow -2 < x < 2$$

centered $x = 0$

10. One of the following series converges to 1.

(a) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1}$ $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ $\frac{1}{2^n}$

(b) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ $\frac{1}{2}, \frac{1}{4}, \dots$

$$\frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = \frac{1}{\frac{1}{2}} = (2)\left(\frac{1}{2}\right)$$

(b) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ converge to 2

(c) $\sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n$. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^{n+1}$

div

(d) $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$ converge to 2

$$\frac{1}{2 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2 + \frac{1}{2}$$

11. The series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

$$\lim \ln\left(\frac{n}{n+1}\right)$$

$$\sum \ln\left(\frac{n}{n+1}\right)$$

conv or Div

(a) Converges to 1.

$$\ln(n) - \ln(n+1)$$

$$\lim \ln(n) - \ln(n+1)$$

(b) Converges to $\ln\left(\frac{1}{2}\right)$.

$$(\ln(1) - \ln(2)) + (\ln(2) - \ln(3)) + \dots + (\ln(n) - \ln(n+1)) + (\ln(n+1) - \ln(n+2))$$

$$\sum \ln(n+1) = \infty \text{ div}$$

(c) Converges to 0.

(d) Diverges.

12. $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \frac{(\ln 2)^3}{3!} + \dots + \frac{(\ln 2)^n}{n!} + \dots =$

$$\frac{(\ln 2)^n}{n!} = \frac{x^n}{n!}$$

(a) $\ln 2$.

$$\frac{x^n}{n!} = e^x$$

$(n+1)$.

$$\frac{x^n}{n!} e^x \times \frac{1}{n+1}$$

(b) $\frac{1}{1-\ln 2}$.

$$e^{1/2} = 2$$

$$\frac{(ln 2)^n}{n!}$$

$$\frac{1}{n!} e^{\ln 2} = 2$$

(c) 2.

(d) The series diverges.

13. The sequence $a_n = n(2^{1/n} - 1)$, $n = 1, 2, 3, \dots$

$$\lim n^{1/n} - 1$$

$$\frac{\ln n}{n}$$

(a) Converges to 0.

$$\frac{1}{n}$$

$$n(\sqrt[n]{2} - 1) \lim \frac{\sqrt[n]{2} - 1}{\frac{1}{n}} = \frac{1}{n}$$

(b) Converges to 1.

$$\frac{1}{n}$$

$$\lim \frac{\sqrt[n]{2} - 1}{\frac{1}{n}} = \lim \frac{1}{n} = 0$$

(c) Converges to $\ln 2$.

$$(\sqrt[n]{2} - 1)^n$$

$$\lim n(\sqrt[n]{2} - 1) \frac{e^n}{e^n} \rightarrow e$$

$$n^{1/n}$$

$$n(\sqrt[n]{2} - 1)$$

$$\lim \sqrt[n]{2^n} - 1$$

$$\lim n \times \lim 0$$

$$\lim n \times \lim 0$$

$$\frac{(n+2)!}{(2n+1)!} \cdot \frac{2^{n+1}}{(n+1)!}$$

$$\lim_{n \rightarrow \infty} \frac{n+2}{2^{n+1}} = \frac{1}{2} < 1$$

14. The series $\sum_{n=1}^{\infty} \frac{(n+1)!}{(2n)!}$

- (a) Converges by ratio test.
- (b) Diverges by ratio test.
- (c) Diverges by nth term test.
- (d) None of the above.

$$\frac{(n+2)!}{(2n+2)!} \cdot \frac{(2n)!}{(n+1)!}$$

15. The series $\sum_{n=2}^{\infty} (-1)^n \frac{\ln(n)}{n}$

- (a) Converges by nth term test.
- (b) Diverges by nth term test.
- (c) Converges absolutely.
- (d) Converges by alternating series test.

$$\frac{\frac{1}{n}}{1} = 0 \quad \frac{(n+2)}{(2n+2)(2n+1)} \rightsquigarrow = 0 \quad \text{com}$$

$$\frac{\ln n}{n} = \frac{\ln(n+1)}{n+1} \rightsquigarrow 0 \quad \text{LT with L'Hopital}$$

$$\frac{(-1)^n \ln(n)}{n^2} \quad \begin{array}{l} \text{if } n \rightarrow \infty \\ \text{then } (-1)^n \rightarrow -1 \end{array} \quad \text{dec}$$

$$\frac{n}{n^2} = \frac{1}{n}$$

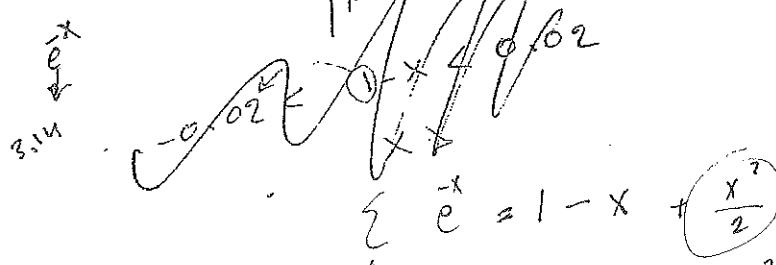
16. The series $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^4-1}}$

- (a) Converges by limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3} \cdot \alpha$.
- (b) Diverges by limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$.
- (c) Converges by direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3} \cdot \alpha$.
- (d) Diverges by nth term test.

$$\frac{n}{\sqrt{n^4-1}} \quad \frac{n^2}{n^2-1} = \frac{1}{1-\frac{1}{n^2}} \quad \sqrt{n^4-1} < \sqrt{n^4} = n^2$$

17. We can approximate e^{-x} by $1-x$ with error less than 0.02 when

- (a) $|x| < 0.4$.
- (b) $|x| < 0.01$.
- (c) $|x| < 0.02$.
- (d) $|x| < 0.2$.



4

$$\frac{x^2}{2} < 0.02$$

$$x^2 < 0.04$$

$$< 0.2$$

$$\frac{n}{\sqrt{n^4-1}} > \frac{1}{n^2}$$

$$\frac{1}{\sqrt{n^4-1}} > \frac{1}{n^2}$$

$$e^{-x} = \frac{x^2}{2!}$$

Question 2(18%) Find the radius and interval of convergence of the series

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{\sqrt{n} \ln n}$$

When does the series converge conditionally, absolutely, diverge?

$$\left| \frac{(x-1)^{n+1}}{\sqrt{n+1} \ln(n+1)} \cdot \frac{\sqrt{n} \ln n}{(x-1)^n} \right| = \left| (x-1) \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \cdot \frac{\ln n}{\ln(n+1)} \right| \quad \lim_{n \rightarrow \infty} \frac{\ln n}{\ln(n+1)} = 1$$

$$= |x-1| \lim_{n \rightarrow \infty} \left| 1 \cdot \left[\frac{\ln n}{\ln(n+1)} \right] \right| = |x-1| \lim_{n \rightarrow \infty} \underbrace{\left| \frac{\ln 1}{\ln n} \right|}_{\text{Zero}}$$

$$|x-1| < 1$$

$$\begin{array}{c} -1 < x-1 < 1 \\ \textcircled{5} \\ \textcircled{2} \quad 0 < x < 2 \end{array}$$

at $x=0 \rightarrow \sum \frac{(-1)^n}{\sqrt{n} \ln n}$, ~~Harmonic Series~~ diverge by nth term test

at $x=2 \rightarrow \sum \frac{(1)^n}{\sqrt{n} \ln n} = \sum \frac{1}{\sqrt{n} \ln n}$
diverge by nth term test

$$\text{radius} = \frac{2+0}{2} = 1$$

Absolutely converge $(0, 2)$

diverge $x \in (-\infty, 0] \cup [2, \infty)$

$$\ln n \sqrt{n} < \frac{1}{\ln n} \quad \frac{1}{\infty} = 0$$

$$\ln n \sqrt{n} > \sqrt{n}$$

$$\cancel{\frac{1}{\ln n \sqrt{n}}} / \cancel{\frac{1}{\sqrt{n}}} = \cancel{\frac{1}{\ln n}}$$

$$\frac{1}{\ln n \sqrt{n}}$$

$$\frac{1}{\ln n \sqrt{n}} \cdot \sqrt{n}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\ln n} = 0$$

$$\cancel{\frac{1}{\ln n \sqrt{n}}} / \cancel{\frac{1}{\sqrt{n}}} = \cancel{\frac{1}{\ln n}}$$

$$\frac{1}{\sqrt{n}} / \frac{1}{\ln n} = \frac{\sqrt{n}}{\ln n}$$

$$= 2\sqrt{n} \cdot \sqrt{n^2}$$

$$= 2\sqrt{n \cdot n^2} = \infty \text{ div}$$

$$= \lim_{n \rightarrow \infty} \sqrt{n^3} = \infty \text{ by nth term test}$$

Question 3(15%) Determine whether the series converge or diverge, justify your answer:

$$(a) \sum_{n=1}^{\infty} \frac{3^n}{3^{n+1} + 4^n}$$

$$\frac{3^{n+1}}{3^{n+1} + 4^{n+1}} < \frac{3^n + 4^n}{3^{n+1} + 4^{n+1}} = 3 \left(\frac{3^n + 4^n}{3^{n+1} + 4^{n+1}} \right)$$

Ratio

~~$\frac{3^n}{3^{n+1} + 4^n}$~~

~~$\frac{3^n}{3^{n+1} + 4^n}$~~

$$= \frac{3^n}{3^{n+1} + 4^{n+1}} + \frac{4^n}{3^{n+1} + 4^{n+1}}$$

$$= \frac{1}{3 + \frac{4^{n+1}}{3^n}}$$

$$\frac{1}{\frac{3^{n+1}}{4^n} + 4}$$

$$= \frac{1}{3 + (\frac{4^n}{3})^4} + \frac{1}{3(\frac{3}{4})^n + 4}$$

1

$$\lim_{n \rightarrow \infty} \frac{1}{3 + (\frac{4^n}{3})^4} + \lim_{n \rightarrow \infty} \frac{1}{3(\frac{3}{4})^n + 4} = \frac{1}{3+0} + \frac{1}{0+4} = \frac{1}{12} + \frac{1}{12} = \frac{2}{12} < 1 \text{ converges}$$

by Ratio test

$$(b) \sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$$

$$\ln a_n = n^2 \ln \left(\frac{n}{n+1} \right)$$

$$\frac{(n+1)^2 \ln \left(\frac{n+1}{n+2} \right)}{n^2 \ln \left(\frac{n}{n+1} \right)} \cdot \frac{1}{\frac{1}{n^2 \ln \left(\frac{n}{n+1} \right)}} = \frac{(n^2 + 2n + 1) \cdot [\ln(n+1) - \ln(n+2)]}{n^2 \cdot [(\ln n) - (\ln(n+1))]} = (2n+1) \frac{\ln \frac{n+1}{n+2}}{\ln \frac{n}{n+1}}$$

$$= 2n+1 \left[\ln \frac{n+1}{n+2} - \ln \frac{n}{n+1} \right] = \frac{\ln \frac{n+1}{n+2}}{\frac{1}{2n+1}} - \frac{\ln \frac{n}{n+1}}{\frac{1}{2n+1}} = \lim_{n \rightarrow \infty} \frac{\infty}{\infty} - \frac{\infty}{\infty}$$

$$\therefore = \frac{\frac{d}{dn} \left(\ln \frac{n+1}{n+2} \right)}{\frac{d}{dn} \left(\frac{1}{2n+1} \right)}$$

$$= \frac{\frac{1}{n+1} \left(\frac{n+1-n}{(n+1)(n+2)^2} \right)}{-\frac{2}{(2n+1)^2}} = \frac{2}{n^2 + 3n + 2} - \frac{2}{n^2 + n} = 2$$

which is finite

$$a_n = e^{\frac{2}{n+1}} \times 1 \\ \text{converges by Ratio}$$

$$(c) \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$$\frac{1}{n \ln n} = \frac{1}{n \ln n} \cdot n = \frac{1}{\ln n} \xrightarrow[n \rightarrow \infty]{\text{L'Hopital}} = \frac{1}{\infty} = 0$$

$$= \frac{1}{\ln n^2} = -\ln(n^2) = -2 \ln n$$

$$= -\frac{\ln n}{\frac{1}{n}} = \frac{\ln n}{\frac{1}{n^2}} = \frac{1}{n} \cdot n^2 = n \xrightarrow[n \rightarrow \infty]{\text{L'Hopital}} \infty$$

diverge by nth term

test

$$= (1-x)^{-2}$$

Question 4 (16%) In this question, you can use $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n, |x| < 1$

(a) Use substitution to show that $\frac{2x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}, |x| < 1$

$$\frac{2x}{1+x^2} = -2x \sum_{n=0}^{\infty} n x^{2n}$$

$$f(a) = 2$$

$$\frac{2x}{1+x^2} = 2x \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$f'(x) = -2x$$

$$\begin{aligned} \frac{2x}{1+x^2} &= 2x \sum_{n=0}^{\infty} (-1)^n (x^2)^n \\ &= \sum_{n=0}^{\infty} (-1)^n 2x \cdot x^{2n} = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}, |x| < 1 \end{aligned}$$

(b) Use (a) to find the Maclaurin series of $\ln(1+x^2)$. $f'(x) = \frac{2x}{1+x^2}$

$\frac{2x}{1+x^2}$ is the first derivative of $\ln(1+x^2)$ \rightarrow

$$\frac{2x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} \quad (\text{by integration is } \int \frac{2x}{1+x^2} dx)$$

$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n 2x^{2n+2}}{2n+2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{2(n+1)} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{n+1}$$

(c) Estimate $\ln(2)$ with error less than 0.1.

