second

Birzeit University- Mathematics Department
Calculus II-Math 132

Spring 2011/2012

Second Exam
Name(Arabic):[redacted]
Number:[redacted]
Instructor of Discussion(Arabic):............
Section:...14...........

Time: 90 Min. Calculators are not allowed. There are 4 questions in 7 pages.

Question 1.(51%) Circle the correct answer:

33
08
02
09
52

1. The sequence $a_n = (1+\frac{1}{n})^{-n}$, $n = 1, 2, 3, \ldots$

$\frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e^{-1}} = e$

   (a) Converges to 1.
   (b) Converges to e.
   (c) Converges to e^{-1}.
   (d) Diverges.

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$

$\left|\frac{(-1)^{n+1}}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}}$ div by p-series

$\frac{(-1)^{n+1}}{\sqrt{n}} \to u_n$ decreas positive $\lim \frac{1}{\sqrt{n}} = 0$

   (a) Converges conditionally.
   (b) Converges absolutely.
   (c) Converges by nth term test.
   (d) Diverges.

3. $\sum_{n=1}^{\infty} \frac{1}{n+\sqrt{n}}$

$n + \sqrt{n} > n$, $> \sqrt{n}$

$\frac{1}{n+\sqrt{n}} = \frac{1}{\infty+\infty} = \frac{1}{\infty} = 0$

   (a) Diverges by nth term test.
   (b) Converges by limit comparison with $\sum_{n=1}^{\infty} \frac{1}{2n}$.
   (c) Diverges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$.
   (d) Diverges by direct comparison with $\sum_{n=1}^{\infty} \frac{1}{2n}$.

4. $\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+1}\right)^n$

   (a) Converges by nth root test.
   (b) Diverges by nth root test.
   (c) Converges by alternating series test.
   (d) None of the above.

5. One of the following statements is always true $\quad (-1, 1, -1, 1)$

(a) A bounded sequence always converges. X →

(b) A monotonic sequence converges. X ~~monotonic + bounded~~ $a_n = n$

(c) A convergent sequence is monotonic. X $\left(1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8} \cdots\right)$

(d) A convergent sequence is bounded.

6. The series $\sum_{n=1}^{\infty} \frac{1}{n^{1/n}}$ $\quad \lim \frac{1}{\sqrt[n]{n}} = \frac{1}{1} = 1$ $\quad \lim \frac{1}{\sqrt[n]{n}}$ $\quad n^{\frac{1}{n}}$ $\quad \frac{1}{1}$

(a) Converges by nth term test.

(b) Converges by nth root test.

(c) Diverges by nth term test.

(d) Diverges by nth root test.

7. The series $\sum_{n=1}^{\infty} (\log_2 x)^n$ converges if $\quad \left(\frac{\ln x}{\ln 2}\right)^n = \left|\frac{n \ln x}{n \ln 2}\right| =$

$\frac{\ln x}{\ln 2}$ $\quad -1 < \frac{\ln x}{\ln 2} < 1$

(a) $x \in (e^{-1}, e)$.

(b) $x \in (\frac{1}{2}, 2)$. ← $\quad -\ln 2 < \ln x < \ln 2$

(c) $x \in (-\frac{1}{2}, \frac{1}{2})$. $\quad e^{-\ln 2} < x < e^{\ln 2}$

(d) $x \in (-1, 1)$.

$-1 < \ln x < 1$ $\quad -\ln 2 < \ln x < \ln 2$

8. One of the following statements is true

(a). If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges. X

(b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ then $\sum a_n$ and $\sum b_n$ both converge or both diverge. X $\quad a_n < b_n$

(c) The alternating harmonic series diverges. X

(d) If $\lim_{n\to\infty} a_n = 1$ then $\sum_{n=1}^{\infty} a_n$ diverges. ✓ nth term test

9. The radius of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ is $\quad -1 < \frac{x}{2} < 1 \quad \left(\frac{x}{2}\right)^n$ $\quad -2 < x < 2$ $\quad \left|\frac{x}{2}\right| < 1$

$\sqrt[n]{\left(\frac{x}{2}\right)^n}$

(a) $R = 1$.

(b) $R = 2$. $\quad -1 < \frac{x}{2} < 1$

(c) $R = 0$. $\quad -1 < \frac{x}{2} < 1 \quad -2 < x < 2$

(d) $R = \infty$. $\quad -2 < x < 2$

center zero

10. One of the following series converges to 1

(a) $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1}$ $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ $\frac{\frac{1}{2}}{1-\frac{1}{2}}$

(b) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$. converge to 2 $\frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{2}{2}-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = (2)\left(\frac{1}{2}\right) = 1$

(c) $\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$. $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^{n+1}$

(d) $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$. converge to 2 $\frac{1}{\frac{2}{2}-\frac{1}{2}}$ $\frac{1}{\frac{1}{2}} = 2 \cdot \frac{1}{2}$

conv or Div

11. The series $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$ $\lim \ln\left(\frac{n}{n+1}\right)$ $\sum_{1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

(a) Converges to 1. $\ln(n) - \ln(n+1)$

(b) Converges to $\ln\left(\frac{1}{2}\right)$. $\lim \ln(n) - \ln(n+1)$

(c) Converges to 0. $(\ln(1) - \ln(2)) + (\ln(2) - \ln(3)) \ldots (\ln(n) - \ln(n)) + (\ln(n) - \ln(n+1))$

(d) Diverges. $\sum \ln(n+1) = \infty$ div

12. $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \frac{(\ln 2)^3}{3!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots =$ $\frac{(\ln 2)^n}{n!} = \frac{x^n}{n!}$

(a) $\ln 2$. $\frac{x^n}{n!} = e^x$

(b) $\frac{1}{1-\ln 2}$.

(c) 2 $\frac{(\ln 2)^n}{n!}$ $\frac{x^n}{n!} = e^x$ $x < 1$ $= e^{\ln 2} = 2$

(d) The series diverges. $e^{\ln 2} = 2$

13. The sequence $a_n = n(2^{1/n} - 1)$, $n = 1, 2, 3, \ldots$ $\lim n\sqrt[n]{2} - n$

(a) Converges to 0. $n\sqrt[n]{2} - n$ $n(\sqrt[n]{2} - 1)$ $\lim \frac{\sqrt[n]{2}}{\frac{1}{n}} - \frac{1}{\frac{1}{n}}$

(b) Converges to 1. $\frac{\sqrt[n]{2}}{\frac{1}{n}}$

(c) Converges to $\ln 2$. $(\sqrt[n]{2} - 1)^n$ $\lim \frac{\sqrt[n]{2}}{\frac{1}{n}} - \lim \frac{1}{\frac{1}{n}} = n$ dx

(d) Diverges. e

$\lim n(\sqrt[n]{2} - 1)$ $\frac{e^{n\sqrt[n]{2}}}{e^n}$ $n\sqrt[n]{2}$ $x^{\frac{1}{n}}$ $n\sqrt[n]{2} - 1$ $\lim \frac{\sqrt[n]{2} - 1}{\frac{1}{n}}$

$\lim n \times \lim 0$

$$\frac{(n+2)!}{(2n+1)!} \cdot \frac{\cancel{2n!}}{\cancel{(n+1)!}}$$

$$\lim \frac{n+2}{2n+1} = \frac{1}{2} < 1$$

14. The series $\sum_{n=1}^{\infty} \frac{(n+1)!}{(2n)!}$

(a) Converges by ratio test.
(b) Diverges by ratio test.
(c) Diverges by nth term test.
(d) None of the above.

$$\frac{(n+2)!}{(2n+2)!} \cdot \frac{(2n)!}{(n+1)!}$$

$$\frac{(n+2)}{(2n+2)(2n+1)} \to = 0 < 1 \text{ conv}$$

$$\frac{\frac{1}{n}}{1} = 0$$

15. The series $\sum_{n=2}^{\infty} (-1)^n \frac{\ln(n)}{n}$

(a) Converges by nth term test.
(b) Diverges by nth term test.
(c) Converges absolutely.
(d) Converges by alternating series test.

$$\frac{\ln n}{n} = \frac{\frac{1}{n}}{1}$$

LCT with $\ln(n)$

$$\text{div} \left| \frac{\ln(n)}{n} \right| \to 0$$

$$\frac{1-\ln(n)}{n^2} \quad 1-\ln(n) = 0 \quad n = e \quad \text{dec}$$

$$\frac{n}{n^2} = \frac{1}{n}$$

16. The series $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^4-1}}$

(a) Converges by limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3}$.
(b) Diverges by limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$.
(c) Converges by direct comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^3}$.
(d) Diverges by nth term test.

$$\frac{n^2}{n^2-1} =$$

$$\sum \frac{n}{\sqrt{n^4-1}} \qquad \frac{n^2}{\sqrt{n^4-1}}$$

$$\sqrt{n^4-1} < \sqrt{n^4}$$

$$\frac{n}{\sqrt{n^4-1}} > \frac{n}{n^2}$$

$$\frac{1}{\sqrt{n^4-1}} > \frac{1}{n}$$

17. We can approximate e^{-x} by $1-x$ with error less than 0.02 when

(a) $|x| < 0.4$.
(b) $|x| < 0.01$.
(c) $|x| < 0.02$.
(d) $|x| < 0.2$.

$$|1-x| < 0.02$$

$$e^x = \frac{x^n}{n!}$$

$$\sum e^{-x} = 1 - x + \frac{x^2}{2}$$

$$\text{error} < \frac{x^2}{2}$$

$$\frac{x^2}{2} \le 0.02$$

$$x^2 \le 0.04$$

$$\le 0.2$$

Question 2(18%) Find the radius and interval of convergence of the series

$$\sum_{n=2}^{\infty} \frac{(x-1)^n}{\sqrt{n}\ln n}$$

When does the series converge conditionally, absolutely, diverge?

$$\left| \frac{(X-1)^{n+1}}{\sqrt{n+1}\,\ln(n+1)} \cdot \frac{\sqrt{n}\,\ln n}{\cancel{(X-1)^{n}}} \right| = \left| (X-1) \cdot \underbrace{\sqrt{\frac{n}{n+1}}}_{\lim = 1} \cdot \underbrace{\frac{\ln n}{\ln(n+1)}}_{} \right|$$

$$= |X-1| \lim_{n\to\infty} \left| 1 \cdot \underbrace{[\ln n - n + 1]}_{} \right| = |X-1| \underbrace{\lim_{n\to\infty} |\ln 1|}_{zero}$$

$|X-1| < 1$

$-1 < X-1 < 1$ (5)

(2) $\boxed{0 < X < 2}$

at $X=0 \rightarrow \sum \frac{(-1)^n}{\sqrt{n}\ln n}$, ~~Alternating Series~~ diverge by nth term test

at $X=2 \rightarrow \sum \frac{(1)^n}{\sqrt{n}\ln n} = \sum \frac{1}{\sqrt{n}\ln n}$

diverge by nth term test

(1) radius $= \frac{2+0}{2} = 1$ (1)

(1) Absolutly converge $x \in (0, 2)$

(1) diverge $x \in (-\infty, 0] \cup [2, \infty)$

$\ln n \sqrt{n} < \frac{1}{\ln n}$ $\quad \frac{1}{\infty} = 0$

$> \ln n$

$\ln n \sqrt{n} > \sqrt{n}$

$\frac{\frac{1}{\ln n \sqrt{n}}}{\frac{1}{\sqrt{n}}}$

$\frac{1}{\ln n \sqrt{n}} \cdot \sqrt{n}$

$\lim_{n\to\infty} \frac{1}{\ln n} = 0$

$\lim \frac{1}{\sqrt{n}\,\ln n}$

$\frac{\frac{1}{\sqrt{n}}}{\ln n} = \frac{2\sqrt{n}}{\frac{1}{n}}$

$= 2\sqrt{n} \cdot \sqrt{n^2}$

$= 2\sqrt{n \cdot n^2}$

$= \lim_{n\to\infty} \sqrt{n^3} = \infty$ div by nth term test

$\sqrt{n}$

$\frac{1}{2\sqrt{}}$

Question 3(15%) Determine whether the series converge or diverge, justify your answer:

(a) $\sum_{n=1}^{\infty} \frac{3^n}{3^n+4^n}$

$$\frac{3}{3^{n+1}+4^{n+1}} \cdot \frac{3^n+4^n}{3^n} = 3\left(\frac{3^n+4^n}{3^{n+1}+4^{n+1}}\right)$$

Ratio

$$= \frac{3^n}{3^{n+1}+4^{n+1}} + \frac{4^n}{3^{n+1}+4^{n+1}}$$

$$= \frac{1}{3+\frac{4^{n+1}}{3^n}} + \frac{1}{\frac{3^{n+1}}{4^n}+4} = \frac{1}{3+\left(\frac{4}{3}\right)^n 4} + \frac{1}{3\left(\frac{3}{4}\right)^n+4}$$

$$\lim_{n\to\infty} \frac{1}{3+\left(\frac{4}{3}\right)^n 4} + \lim \frac{1}{3\left(\frac{3}{4}\right)^n+4} = \frac{1}{3+0} + \frac{1}{0+4} = \frac{4}{12} + \frac{3}{12} = \frac{7}{12} < 1$$ converg

by Ratio test

(b) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

$$\ln a_n = n^2 \ln\left(\frac{n}{n+1}\right)$$

$$(n+1)^2 \ln\left(\frac{n+1}{n+2}\right) \cdot \frac{1}{n^2 \ln\left(\frac{n}{n+1}\right)} = \frac{(\cancel{n^2}+2n+1).\left[\ln(n+1) - \ln(n+2)\right]}{\cancel{n^2}\left[(\ln n) - (\ln n+1)\right]} = (2n+1)\frac{\ln\frac{n+1}{n+2}}{\ln\frac{n}{n+1}}$$

$$= 2n+1\left[\ln\frac{n+1}{n+2} - \ln\frac{n}{n+1}\right] = \frac{\ln\frac{n+1}{n+2}}{\frac{1}{2n+1}} - \frac{\ln\frac{n}{n+1}}{\frac{1}{2n+1}} = \lim \frac{\infty}{\infty} - \frac{\infty}{\infty}$$

$$= \frac{\frac{n+2}{n+1}\left(\frac{1}{(n+2)^2}\right)}{\frac{1}{2}} - \frac{\frac{n+1}{n}\left(\frac{1}{(n+1)^2}\right)}{\frac{1}{2}} = \frac{2}{n^2+3n+2} - \frac{2}{n^2+n} = 2$$

$\infty - \infty = 0$

إكمال الحل خلف الصفحة

$a_n = e^2 < 1$ converge by Ratio

(c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

$$\cancel{\frac{\frac{1}{n\ln n}}{\frac{1}{n}} = \frac{1}{n\ln n} \cdot n = \frac{1}{\ln n} \lim_{n\to\infty} = \frac{1}{\infty} = \text{zero}}$$

$$= \frac{1}{\ln n^n} = -\ln(n^n) = -n\ln n$$

$$= -\frac{\ln n}{\frac{1}{n}} \overset{\text{L'Hôpital}}{=} \frac{\frac{1}{n}}{\frac{1}{n^2}} = \frac{1}{n} \cdot n^2 = n \lim_{n\to\infty} \infty$$

diverge by nth term test

$-(1-x)^{-2}$

Question 4(16%) In this question, you can use $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$.

(a) Use substitution to show that $\frac{2x}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}$, $|x| < 1$

$f(0) = 2$

$f'(0) = -2x$

4

$(2x)\frac{1}{1-(-x^2)} = 2x\sum_{n=0}^{\infty}(-x^2)^n = 2x\sum_{n=0}^{\infty}(-1)^n(x^{2n})$

$= \sum_{n=0}^{\infty}(-1)^n 2x \cdot x^{2n} = \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}$, $|x|<1$

(b) Use (a) to find the Maclaurin series of $\ln(1+x^2)$. $f'(x) = \frac{2x}{1+x^2}$

$\frac{2x}{1+x^2}$ is the first derivative of $\ln(1+x^2)$ ∴

$\frac{2x}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n 2x^{2n+1}$ (by integration ~~term by term~~)

$\ln(1+x^2) = \sum_{n=0}^{\infty}(-1)^n\frac{2x^{2n+2}}{2n+2} = \sum_{n=0}^{\infty}(-1)^n\frac{\cancel{2}x^{2n+2}}{\cancel{2}(n+1)} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+2}}{n+1} + C$

5

(c) Estimate $\ln(2)$ with error less than 0.1.

$\int\frac{1}{1+x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{}$

$\ln x$ $\frac{1}{x}$ $\ln(1+x)$

$\ln x$

$\ln x = \frac{1}{x} = \frac{-1}{x^2} + 1$

0